

Transversal Twistor Spaces of Foliations

by
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ABSTRACT. The transversal twistor space of a foliation \mathcal{F} of an even codimension is the bundle $\mathcal{Z}(\mathcal{F})$ of the complex structures of the fibers of the transversal bundle of \mathcal{F} . On $\mathcal{Z}(\mathcal{F})$ there exists a foliation $\hat{\mathcal{F}}$ by covering spaces of the leaves of \mathcal{F} , and any Bott connection of \mathcal{F} produces an ordered pair $(\mathcal{I}_1, \mathcal{I}_2)$ of transversal almost complex structures of $\hat{\mathcal{F}}$. The existence of a Bott connection which yields a structure \mathcal{I}_1 that is projectable to the space of leaves is equivalent to the fact that \mathcal{F} is a transversally projective foliation. A Bott connection which yields a projectable structure \mathcal{I}_2 exists iff \mathcal{F} is a transversally projective foliation which satisfies a supplementary cohomological condition, and, in this case, \mathcal{I}_1 is projectable as well. \mathcal{I}_2 is never integrable. The essential integrability condition of \mathcal{I}_1 is the flatness of the transversal projective structure of \mathcal{F} .

The twistor construction [12, 1, 11] associates (almost) complex manifolds to differentiable manifolds endowed with a connection, and, in particular, to Riemannian manifolds with the Levi-Civita connection. In some cases, this allows to associate objects of Riemannian (conformal) geometry to holomorphic objects of the twistor space. Our aim is to give a similar construction which will associate a transversally (almost) complex foliation to a given differentiable foliation. We will define a twistor space which consists of all the complex structures of the fibers of the transversal bundle of the foliation. There exists a well defined lift of the original foliation to the twistor space, the leaves of which are covering spaces of the original leaves. Then, a Bott

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connection of the original foliation defines two transversal almost complex structures of the lifted foliation, and we study the conditions which ensure that these structures are projectable to the space of leaves, then, the integrability conditions.

The projectability problem does not exist for twistor spaces of manifolds. We show that Bott connections which yield projectable structures exist iff the foliation has a transversal projective structure which satisfies a certain cohomological condition. The general transversally projective foliations are exactly the foliations which admit a torsionless Bott connection such that a specified one of the two almost complex structures of the twistor space mentioned above is projectable.

The integrability conditions are analogous to those of the classical twistor spaces e.g., [11]. In particular, the essential integrability condition of the projectable almost complex structures transversal to the lift of a transversally projective foliation is the flatness of the projective transversal structure of the foliation. Equivalently, the flatness condition means that the foliation has a Haefliger cocycle which consists of projective transformations expressed in non homogeneous coordinates.

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1 Recall on foliations

We assume that the reader is familiar with the differential geometry of foliations, as exposed for instance in [8], and only recall a few things which we particularly need. Throughout the paper, *differentiable* means of class C^∞ , and all the involved objects are of this class. We are looking at a given *foliated manifold* (M, \mathcal{F}) where M is a manifold of dimension $n = p + s$, and \mathcal{F} is a foliation of M , of codimension s . We denote by F the tangent bun-

dle $T\mathcal{F}$, and we identify the transversal bundle $N\mathcal{F} = TM/F$ with a fixed complementary subbundle E (i.e., $TM = E \oplus F$), by sending each vector $Z \in E_x$ to the class $[Z]_F = \{Z + F_x\} \in N_x\mathcal{F}$ ($x \in M$). In principle, one is interested in constructions which depend on $N\mathcal{F}$, and not on the choice of E but, the choice of E is technically helpful since it allows for a natural bigrading of differential forms and tensor fields. Our convention is that if a bidegree is (u, v) then u is the E -degree and v is the (leafwise) F -degree.

The most important fact is the existence of a decomposition of the exterior differential d namely,

$$(1.1) \quad d = d'_{(1,0)} + d''_{(0,1)} + \partial_{(2,-1)},$$

where the indices denote the bidegrees of the operators. The operator d'' essentially is the exterior differential along the leaves of \mathcal{F} , and yields the \mathcal{F} -tangential de Rham cohomology, which is that of the sheaf $\Phi(M, \mathcal{F})$ of germs of functions on M which are constant along the leaves of \mathcal{F} . On the other hand, a differential form of type $(u, 0)$ will be called a *transverse form*.

It will be our general convention to say that a geometric object on M is \mathcal{F} -foliated or \mathcal{F} -projectable (the two terms will be used interchangeably) if it projects to the local *slice spaces* of \mathcal{F} . For instance, $f \in C^\infty(M)$ is foliated if it is constant along the leaves of \mathcal{F} i.e., $d''f = 0$, a mapping $\varphi : (M_1, \mathcal{F}_1) \rightarrow (M_2, \mathcal{F}_2)$ is foliated if it sends leaves of \mathcal{F}_1 into leaves of \mathcal{F}_2 . A differential form λ on M is foliated if it is the pullback of local forms of the slice spaces (hence, λ also is a transverse form). A vector field X on M is foliated if it is projectable to the slice spaces or, equivalently, its flow preserves \mathcal{F} (see, for instance, [8, 13, 14] for details).

A principal bundle $P \rightarrow M$ of structure group G is *foliated* if it is endowed with a *foliated structure*, i.e., a maximal local trivialization atlas with foliated transition functions. A bundle may have many foliated structures or none. Any associated bundle of a principal foliated bundle is again called a *foliated bundle*. This definition applies to vector bundles, in particular. It is an immediate consequence of the definition that any foliated bundle over (M, \mathcal{F}) has a canonical foliation $\tilde{\mathcal{F}}$, the leaves of which are covering spaces of the leaves of \mathcal{F} . For the principal bundle P , this amounts to the definition given in [4]. Finally, cross sections of a foliated bundle are foliated if they send leaves of \mathcal{F} to leaves of $\tilde{\mathcal{F}}$.

If P is a foliated principal bundle over (M, \mathcal{F}) , an \mathcal{F} -adapted connection on P (*Bott connection* [2]) is a connection defined by a \mathcal{G} -valued connection

form, where \mathcal{G} is the Lie algebra of G , which is of the $\tilde{\mathcal{F}}$ -type $(1, 0)$ (i.e., a transverse form). Equivalently, the restricted holonomy of the connection along paths in the leaves of \mathcal{F} is trivial. Furthermore, a connection which is defined by a foliated connection form is a *projectable connection* (details in [4, 8]).

A notion which will be important for us is that of a *projectable (foliated) distribution*. By definition, a tangent distribution D of M is projectable if $F \subseteq D$, and, locally, D is projection-related with distributions of the slice spaces. We have

1.1 Proposition. *Let D be a distribution which includes F . Then D is projectable iff it satisfies one of the following equivalent conditions: a) there exists an open covering of M , $M = \cup_{\alpha} U_{\alpha}$ such that $D|_{U_{\alpha}}$ is spanned by foliated vector fields $\{X_k\}$; b) there exists an open covering $M = \cup_{\alpha} U_{\alpha}$ such that $D|_{U_{\alpha}}$ has equations $\xi_{(\alpha)}^h = 0$, where $\xi_{(\alpha)}^h$ are $(1, 0)$ -forms which satisfy the condition*

$$(1.2) \quad d''\xi_{\alpha}^h = 0 \quad (\text{mod. } \xi_{\alpha}^h).$$

Proof. The equivalence of condition a) with the definition is obvious. For condition b), and if $\dim D = m \geq p$, it suffices to consider independent forms ξ^h , $h = 1, \dots, s + p - m$. If D is projectable, on adapted coordinate neighborhoods U_{α} of \mathcal{F} (e.g., [8]), D has equations $\xi_{(\alpha)}^h = 0$, which are pullbacks of the equations of the projection of D onto U_{α}/\mathcal{F} . Then, the forms $\xi_{(\alpha)}^h$ are foliated, and (1.2) holds.

Conversely, if D has equations which satisfy (1.2), any equivalent equations $\eta^l = 0$ ($l = 1, \dots, s + p - m$) where $\eta^l = \sum_h A_h^l \xi^h$ also satisfy (1.2). In particular, we may use a system of the form

$$(1.3) \quad \eta^l = \sigma^l + \sum_{u=s+p-m+1}^s f_u^l \sigma^u,$$

where (σ^l, σ^u) is a foliated basis of E^* , the dual bundle of the transversal bundle E of F . Then (1.2) for η^l means

$$(1.4) \quad d''\eta^l = \sum_u d''f_u^l \wedge \sigma^u = \sum_h \eta^h \wedge \varphi_h^l,$$

where φ_h^l are some $(0, 1)$ -forms. But, the presence of the terms σ^l in (1.3) make the last equality (1.4) impossible unless $\varphi_h^l = 0$. Therefore, η^l are foliated forms, and D is foliated. Q.e.d.

2 Transversal twistor spaces

In this section, \mathcal{F} is a foliation of an even codimension $s = 2q$, $q \geq 1$, on a manifold M^n ($n = p + 2q$). Similarly to the classical theory [11], we define the *transversal twistor space*

$$(2.1) \quad \begin{aligned} \mathcal{Z}(\mathcal{F}) &= \{J_x \in \text{End}(N_x \mathcal{F}) \mid J^2 = -Id, x \in M\} \\ &\approx \{J_x \in \text{End } E_x \mid J^2 = -Id, x \in M\}, \end{aligned}$$

where the equivalence follows from the identification $N\mathcal{F} \approx E$ defined in Section 1. We will study the structure of this space by using a moving frame method similar to that of [3, 17]. We express everything in the E -version of (2.1) but, the results may be transposed to the $N\mathcal{F}$ -version.

Denote by $E^c = E \otimes_{\mathbf{R}} \mathbf{C}$ the complexification of E , and consider bases (frames) of the fibers E_x^c ($x \in M$) which are of the form $B = (b, \bar{b})$, where b is a line matrix of q vectors $(b_i)_{i=1}^q$, and the bar denotes complex conjugation. (Our convention for matrix entries is: the upper index is the line index and the lower index is the column index.) Such bases are obtained by considering q -planes $S \subset E^c$ of *real index* 0 (for $S \subset E^c$, the real index is defined as the dimension of $S \cap \bar{S}$), and taking bases b of S . Bases B as above will be called *non-real bases (frames)*.

We denote by $\mathcal{B}(\mathcal{F})$ the $Gl(2q, \mathbf{R})$ -principal bundle of all the non-real frames of the fibers of E^c , where overall in this paper

$$(2.2) \quad Gl(2q, \mathbf{R}) = \left\{ \Phi = \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \mid A, B \in \mathcal{M}(q, q; \mathbf{C}), \det \Phi \neq 0 \right\},$$

where $\mathcal{M}(i, j; \mathbf{C})$ is the space of complex (i, j) -matrices. This obviously is a foliated bundle with the foliated structure defined by that of $N\mathcal{F}$, which is known to be a foliated vector bundle [8]. Therefore, $\mathcal{B}(\mathcal{F})$ has the lifted foliation $\tilde{\mathcal{F}}$ noticed in Section 1.

Now, we note the existence of the natural projection

$$(2.3) \quad P : \mathcal{B}(\mathcal{F}) \rightarrow \mathcal{Z}(\mathcal{F})$$

defined by $P(b, \bar{b}) = J$ where J is the complex structure with the $(\sqrt{-1})$ -eigenspace $\text{span}\{b_i\}$. (2.3) is a principal fibration of structure group

$$(2.4) \quad Gl(q, \mathbf{C}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} \mid A \in \mathcal{M}(q, q; \mathbf{C}), \det A \neq 0 \right\}.$$

The natural projection

$$(2.5) \quad Q : \mathcal{Z}(\mathcal{F}) \rightarrow M$$

is a bundle associated with $\pi : \mathcal{B}(\mathcal{F}) \rightarrow M$, with the fiber equal to the homogeneous space $Gl(2q, \mathbf{R})/Gl(q, \mathbf{C})$ of the real dimension $2q^2$. In particular, $\mathcal{Z}(\mathcal{F})$ is a differentiable manifold of dimension $p + 2q(q + 1)$. Moreover, $\mathcal{Z}(\mathcal{F})$ admits a lift of the foliation \mathcal{F} which will be denoted by $\hat{\mathcal{F}}$, and the leaves of $\hat{\mathcal{F}}$ are covering spaces of the leaves of \mathcal{F} .

We discuss geometric structures on $\mathcal{Z}(\mathcal{F})$ by using some natural local cotangent bases.

We start with the principal bundle $\mathcal{B}(M, \mathcal{F})$ of the tangent bases of M which are of the form (B, C) , where C is a real basis of F and B is a non-real basis of E^c (*transversally non-real, \mathcal{F} -adapted bases*). The structure group of $\mathcal{B}(M, \mathcal{F})$ is

$$(2.6) \quad G := \left\{ \begin{pmatrix} \mathcal{X} & 0 \\ 0 & \mathcal{S} \end{pmatrix} / \mathcal{X} \in Gl(2q, \mathbf{R}), \mathcal{S} \in Gl(p, \mathbf{R}) \right\},$$

and there exists a principal fibration

$$(2.7) \quad K : \mathcal{B}(M, \mathcal{F}) \rightarrow \mathcal{B}(\mathcal{F})$$

of group

$$(2.8) \quad G_0 := \left\{ \begin{pmatrix} Id & 0 \\ 0 & \mathcal{S} \end{pmatrix} \right\} \subseteq G,$$

where $K(B, C) = B$.

On $\mathcal{B}(M, \mathcal{F})$, we have the *canonical 1-form* θ [6], where, for $\Xi \in T_{(B,C)}\mathcal{B}(M, \mathcal{F})$, $\theta(\Xi)$ is the column of the components of $\pi_*(\Xi)$ with respect to the basis (B, C) , $\pi : \mathcal{B}(M, \mathcal{F}) \rightarrow M$ again being the natural projection. We will denote

$$(2.9) \quad \theta = \begin{pmatrix} \beta \\ \bar{\beta} \\ \gamma \end{pmatrix},$$

where β is a column (β^i) of complex valued 1-forms and γ is a column (γ^u) of real valued 1-forms, $i = 1, \dots, q$; $u = 1, \dots, p$. If $g \in G$, the right translation R_g on $\mathcal{B}(M, \mathcal{F})$ acts on θ by [6]

$$(2.10) \quad R_g^* \theta = g^{-1} \circ \theta.$$

In particular, if $g \in G_0$ defined by (2.8), β^i and $\bar{\beta}^i$ are invariant by R_g , and their pullbacks $s^*\beta^i$, $s^*\bar{\beta}^i$ by a system $\{s\}$ of trivializing local cross sections of K yield global 1-forms, again denoted $\beta^i, \bar{\beta}^i$ on $\mathcal{B}(\mathcal{F})$. Hereafter, it will be our general convention not to write the pullback mappings, letting the context indicate the manifolds where the forms are to be considered.

We also notice that $\tilde{\beta} := \begin{pmatrix} \beta \\ \bar{\beta} \end{pmatrix}$ coincides with the transversal canonical 1-form defined in [8], and that the right translation R_h of $\mathcal{B}(\mathcal{F})$, $h \in Gl(2q, \mathbf{R})$ acts on $\tilde{\beta}$ by

$$(2.11) \quad R_h^* \tilde{\beta} = h^{-1} \circ \tilde{\beta}.$$

For γ of (2.9), if $g \in G_0$, then

$$(2.12) \quad R_g^* \gamma = \mathcal{S}^{-1} \circ \gamma.$$

The forms $\beta^i, \bar{\beta}^i, \gamma^u$ (i.e., $s^*\beta^i, s^*\bar{\beta}^i, s^*\gamma^u$ for local trivializations s of K) will be a part of a local basis of $T^*\mathcal{B}(\mathcal{F})$. A complete basis of $T^*\mathcal{B}(\mathcal{F})$ may be obtained by adding an adapted connection form ϖ on $\mathcal{B}(\mathcal{F})$ with values in the Lie algebra $gl(2q, \mathbf{R})$ associated with the representation (2.2). With the natural symmetric decomposition [11]

$$(2.13) \quad gl(2q, \mathbf{R}) = gl(q, \mathbf{C}) \oplus \mathcal{N},$$

and using (q, q) -blocks, we will write

$$(2.14) \quad \varpi = \begin{pmatrix} \omega & \theta \\ \bar{\theta} & \bar{\omega} \end{pmatrix} = \begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix} + \begin{pmatrix} 0 & \theta \\ \bar{\theta} & 0 \end{pmatrix}.$$

The form of the matrices in the right hand side of (2.14) is characteristic for the two terms of the decomposition (2.13), and provides the definition of these two terms.

The reason for taking an adapted connection is that, in this case, the lifted foliation $\tilde{\mathcal{F}}$ of $\mathcal{B}(\mathcal{F})$ has the equations

$$(2.15) \quad \beta = 0, \bar{\beta} = 0, \varpi = 0,$$

and the horizontal distribution of the connection ϖ contains $T\tilde{\mathcal{F}}$.

Furthermore, we recall that [6]

$$(2.16) \quad R_h^* \varpi = ad h^{-1} \varpi \quad (h \in Gl(2q, \mathbf{R})).$$

If $h \in Gl(q, \mathbf{C})$ is given by the matrix (2.4), (2.16) reduces to

$$(2.17) \quad R_h^* \omega = A^{-1} \omega A, \quad R_h^* \theta = A^{-1} \theta \bar{A}.$$

The interpretation of the formulas (2.17) for the principal fibration P of (2.3) is: (i) $\begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix}$ is a connection of P , called the *associated connection* of ϖ ; (ii) $\begin{pmatrix} 0 & \theta \\ \bar{\theta} & 0 \end{pmatrix}$ is a horizontal, tensorial 1-form [6] called the *associated tensorial form* of the fibration P .

Notice also that, if $h \in Gl(q, \mathbf{C})$ the relation (2.11) becomes

$$(2.18) \quad R_h^* \beta = A^{-1} \beta, \quad R_h^* \bar{\beta} = \bar{A}^{-1} \bar{\beta}.$$

We summarize the results in

2.1 Proposition. *i). The pullbacks of the entries of the matrices $(\gamma, \beta, \bar{\beta}, \theta, \bar{\theta})$ of (2.9), (2.14) by the (not explicitly written) local cross sections of the fibration P of (2.3) are local bases of the cotangent bundle $T^* \mathcal{Z}(\mathcal{F})$. ii). The local equations of the lifted foliation $\hat{\mathcal{F}}$ of \mathcal{F} to $\mathcal{Z}(\mathcal{F})$ are*

$$(2.19) \quad \beta = 0, \bar{\beta} = 0, \theta = 0, \bar{\theta} = 0,$$

and the equations

$$(2.20) \quad \gamma = 0$$

yield a well defined complementary subbundle \hat{E} of $\hat{F} = T\hat{\mathcal{F}}$ i.e., $T\mathcal{Z}(\mathcal{F}) = \hat{E} \oplus \hat{F}$. iii). There exists a decomposition $T\mathcal{Z}(\mathcal{F}) = \mathcal{H} \oplus \mathcal{V}$, where \mathcal{V} is the $2q^2$ -dimensional vertical distribution, tangent to the fibers of Q (see (2.5)), and \mathcal{H} is the $(p + 2q)$ -dimensional horizontal distribution given by the projection by P of the horizontal distribution of ϖ . The terms of this decomposition have the following equations

$$(2.21) \quad (\mathcal{H}) \quad \theta = 0, \bar{\theta} = 0; \quad (\mathcal{V}) \quad \beta = 0, \bar{\beta} = 0, \gamma = 0,$$

and $\hat{F} \subseteq \mathcal{H}$.

We must emphasize that the cobases defined in Proposition 2.1 depend on the choice of the adapted connection ϖ . A change of this connection is of the form [6]

$$(2.22) \quad \varpi' = \varpi + \begin{pmatrix} \xi & \zeta \\ \bar{\zeta} & \bar{\xi} \end{pmatrix},$$

where the additional term is a $gl(2q, \mathbf{R})$ -valued, horizontal, tensorial 1-form on $\mathcal{B}(\mathcal{F})$, which satisfies (2.16). Hence, in the new cobasis we have the same $\gamma, \beta, \bar{\beta}$, but,

$$(2.23) \quad \theta' = \theta + \zeta, \quad \bar{\theta}' = \bar{\theta} + \bar{\zeta},$$

where ζ satisfies the second condition (2.17), and vanishes on \mathcal{V} .

3 Almost complex structures on $\mathcal{Z}(\mathcal{F})$

By Proposition 2.1 i), connection dependent geometric structures of the twistor space $\mathcal{Z}(\mathcal{F})$ may be discovered by looking for geometric objects defined by the cobases $(\gamma, \beta, \bar{\beta}, \theta, \bar{\theta})$ which are invariant by transformations of the form (2.12), (2.17), (2.18). Then, the effect of a change of the connection may be studied by using (2.23).

Following usual twistor theory, as in [3] for instance, we notice the existence of the following well defined subbundles of the complexification \hat{E}^c of the $\hat{\mathcal{F}}$ -transversal bundle \hat{E} of Proposition 2.1

$$(3.1) \quad (\mathcal{C}_1) \quad \gamma = 0, \beta = 0, \theta = 0,$$

$$(3.2) \quad (\mathcal{C}_2) \quad \gamma = 0, \beta = 0, \bar{\theta} = 0.$$

Obviously, these are subbundles of real index zero hence, there exist two well defined, transversal, almost complex structures of $\hat{\mathcal{F}}$, say $\mathcal{I}_1, \mathcal{I}_2$, which have $\mathcal{C}_1, \mathcal{C}_2$ as their $(-\sqrt{-1})$ -eigenspace, respectively (the sign was chosen to agree with [3]). In the $N\mathcal{F}$ -version of (2.1) $\mathcal{C}_1, \mathcal{C}_2$ become the subbundles of $N^c\hat{\mathcal{F}} := N\hat{\mathcal{F}} \otimes \mathbf{C}$ which have the annihilator

$$(3.1') \quad \beta = 0, \theta = 0,$$

$$(3.2') \quad \beta = 0, \bar{\theta} = 0,$$

respectively, in $(N^c\hat{\mathcal{F}})^* \subseteq T^*\mathcal{Z}(\mathcal{F})$, and the structures $\mathcal{I}_1, \mathcal{I}_2$ of $N\hat{\mathcal{F}}$ depend on the connection ϖ but, they do not depend on the choice of E .

For a later use, let us also notice that the equations (3.1'), (3.2') define the distributions $\mathcal{C}'_1 := F \oplus \mathcal{C}_1$, $\mathcal{C}'_2 := F \oplus \mathcal{C}_2$, respectively, on $\mathcal{Z}(\mathcal{F})$.

We will study conditions which ensure that $\hat{\mathcal{F}}$ is a transversally holomorphic foliation on $\mathcal{Z}(\mathcal{F})$. A first step, which is specific for foliations, and is not encountered in twistor theory on manifolds, is to find the conditions for $\mathcal{I}_1, \mathcal{I}_2$ to be $\hat{\mathcal{F}}$ -projectable.

In order to formulate and prove the result we need some notation and preparations. We will denote by ∇ the covariant derivative defined on E by ϖ , as well as the equivalent covariant derivative defined on $N\mathcal{F}$ by the identification $N\mathcal{F} \approx E$, by R_∇ the corresponding curvature operator, and by

$$T_\nabla(X, Y) := \nabla_X(\pi_E Y) - \nabla_Y(\pi_E X) - \pi_E[X, Y] \quad (X, Y \in \Gamma TM),$$

where $\pi_E : TM \rightarrow E$ is the projection of the splitting $TM = F \oplus E$ and Γ denotes the space of global cross sections of a vector bundle, the torsion [6, 8]. Furthermore, we may define the *Ricci tensor* of an adapted connection by

$$Ric_{(\nabla, E)}(X, Y) = tr[Z \mapsto R_\nabla(Z, X)\pi_E Y] \quad (X, Y \in T_x M, Z \in E_x, x \in M),$$

This Ricci tensor depends on the choice of E , except if either $Y \in F_x$, hence $Ric_{(\nabla, E)}(X, Y) = 0$, or $X \in F_x$, and then

$$Ric_{(\nabla, E)}(X, Y) = tr[\pi_N Z \mapsto R_\nabla(Z, X)\pi_N Y] \quad (X, Y, Z \in T_x M),$$

where $\pi_N : TM \rightarrow \mathcal{F}$ is the natural projection. In these cases, we will denote the tensor by Ric_∇ .

The *structure equations* of the connection ϖ [6, 8] are

$$(3.3) \quad \begin{pmatrix} d\beta \\ d\bar{\beta} \end{pmatrix} = - \begin{pmatrix} \omega & \theta \\ \bar{\theta} & \bar{\omega} \end{pmatrix} \wedge \begin{pmatrix} \beta \\ \bar{\beta} \end{pmatrix} + \begin{pmatrix} T \\ \bar{T} \end{pmatrix},$$

$$(3.4) \quad \begin{pmatrix} d\omega & d\theta \\ d\bar{\theta} & d\bar{\omega} \end{pmatrix} = - \begin{pmatrix} \omega & \theta \\ \bar{\theta} & \bar{\omega} \end{pmatrix} \wedge \begin{pmatrix} \omega & \theta \\ \bar{\theta} & \bar{\omega} \end{pmatrix} + \begin{pmatrix} \Omega & \Theta \\ \bar{\Theta} & \bar{\Omega} \end{pmatrix},$$

where the last term of (3.3) consists of the *torsion forms*, and the last term of (3.4) consists of the *curvature forms* of the connection. The forms in (3.3), (3.4) are global forms on $\mathcal{B}(\mathcal{F})$. Since an adapted connection has no restricted leafwise holonomy (Section 1), the curvature forms vanish if evaluated on two

arguments tangent to $\tilde{\mathcal{F}}$. The torsion forms vanish if at least one argument is tangent to $\tilde{\mathcal{F}}$.

By an exterior differentiation of (3.3) and (3.4), one deduces the *Bianchi identities*. In particular, from the Bianchi identity provided by (3.3) we get

$$(3.5) \quad R_{\nabla}(X, Y)Z - R_{\nabla}(Z, Y)X = \nabla_Y T_{\nabla}(X, Z) \quad (Y \in F_x, X, Z \in E_x),$$

with the corresponding $N\mathcal{F}$ -version

$$(3.5') \quad \begin{aligned} R_{\nabla}(X, Y)\pi_N Z - R_{\nabla}(Z, Y)\pi_N X \\ = \nabla_Y T_{\nabla}(X, Z) \quad (Y \in F_x, X, Z \in T_x M). \end{aligned}$$

We will denote by $i(X)\nabla_Y T_{\nabla}$ the endomorphism $Z \mapsto \nabla_Y T_{\nabla}(X, Z)$ of E , and notice the following consequence of (3.5')

$$(3.5'') \quad Ric_{\nabla}(Y, X) = tr[R_{\nabla}(X, Y) - i(X)\nabla_Y T_{\nabla}] \quad (Y \in F_x, X \in T_x M).$$

Now, we can state the following results

3.1 Theorem. 1). The $\hat{\mathcal{F}}$ -transversal almost complex structure \mathcal{I}_1 is projectable iff either i) $q \geq 2$, and $\forall x \in M, \forall Y \in F_x, \forall X, Z \in T_x M$, one has

$$(3.6)_1 \quad R_{\nabla}(X, Y)\pi_N Z = \lambda(X, Y)\pi_N Z + \mu(Z, Y)\pi_N X,$$

where, necessarily, the coefficients λ, μ are given by

$$(3.6')_1 \quad \begin{aligned} \lambda(X, Y) &= \frac{1}{4q^2 - 1}(2q \operatorname{tr} R_{\nabla}(X, Y) - Ric_{\nabla}(Y, X)) \\ &= \frac{1}{2q + 1} \operatorname{tr} R_{\nabla}(X, Y) + \frac{1}{4q^2 - 1} \operatorname{tr} i(X)\nabla_Y T_{\nabla}, \end{aligned}$$

$$(3.6'')_1 \quad \begin{aligned} \mu(X, Y) &= \frac{1}{1 - 4q^2}(\operatorname{tr} R_{\nabla}(X, Y) - 2q Ric_{\nabla}(Y, X)) \\ &= -\frac{1}{2q + 1} \operatorname{tr} R_{\nabla}(X, Y) - \frac{2q}{4q^2 - 1} \operatorname{tr} i(X)\nabla_Y T_{\nabla} \end{aligned}$$

or ii) $q = 1$, and

$$(3.6)_2 \quad R_{\nabla}(X, Y)\pi_N X = \nu(X, Y)\pi_N X,$$

where X, Y are as above and necessarily

$$(3.6')_2 \quad \begin{aligned} \nu(X, Y) &= \frac{1}{3}[\text{tr } R_\nabla(X, Y) + \text{Ric}_\nabla(Y, X)] \\ &= \frac{2}{3}\text{tr } R_\nabla(X, Y) - \frac{1}{3}\text{tr } i(X)\nabla_Y T_\nabla. \end{aligned}$$

2). For all q , the structure \mathcal{I}_2 is projectable iff

$$(3.6)_3 \quad R_\nabla(X, Y) = \alpha(X, Y)Id,$$

where Id is the identity mapping of $N\mathcal{F}$ and, necessarily

$$(3.6')_3 \quad \alpha(X, Y) = \frac{1}{2q}\text{tr } R_\nabla(X, Y),$$

and in this case the structure \mathcal{I}_1 must also be projectable.

Proof. While the results are formulated for $N\mathcal{F}$, for the proof we choose a transversal bundle E , and take $X, Z \in E_x$, forgetting about the π_N . Generally, if (V, \mathcal{S}) is a foliated manifold, projectability of an almost complex structure J of $N\mathcal{S}$ onto the local slice spaces, is equivalent to the fact that for any foliated cross section s of $N\mathcal{S} \approx E$ (E is transversal to $T\mathcal{S}$), J_s is also foliated, as well as to the fact that, if $\mathcal{C} \subseteq E^c$ is the $(-\sqrt{-1})$ -eigendistribution of J , then $\mathcal{C}' := T\mathcal{S} \oplus \mathcal{C}$ is projectable. In our case, we will discuss this property for the distributions $\mathcal{C}'_1, \mathcal{C}'_2$ defined by (3.1'), (3.2').

Since the curvature and torsion forms are horizontal on the principal bundle $\mathcal{B}(\mathcal{F})$, we may write the entries of the torsion and curvature matrices as follows

$$(3.7) \quad T^a = \frac{1}{2}T_{bc}^a \beta^b \wedge \beta^c + T_{b\bar{c}}^a \beta^b \wedge \beta^{\bar{c}} + \frac{1}{2}T_{b\bar{c}}^a \beta^{\bar{b}} \wedge \beta^{\bar{c}},$$

$$(3.8) \quad \begin{aligned} \Theta_b^a &= \frac{1}{2}R_{bcd}^a \beta^c \wedge \beta^d + R_{b\bar{c}d}^a \beta^c \wedge \beta^{\bar{d}} + \frac{1}{2}R_{b\bar{c}d}^a \beta^{\bar{c}} \wedge \beta^{\bar{d}} \\ &\quad + R_{bcu}^a \beta^c \wedge \gamma^u + R_{b\bar{c}u}^a \beta^{\bar{c}} \wedge \gamma^u, \end{aligned}$$

etc. In formulas (3.7), (3.8), we use the Einstein summation convention, $a, b, c, d = 1, \dots, q$, and $u = 1, \dots, p$. The coefficients, except those in the last two terms of (3.8), are global functions on $\mathcal{B}(\mathcal{F})$. The indices in the left

hand side of (3.8) are in agreement with the position of the block Θ in the curvature matrix.

If we pull back equations (3.3), (3.4), (3.7), (3.8) by local cross sections s of P of (2.3), we get the same equations on $\mathcal{Z}(\mathcal{F})$, locally. The terms of (3.7), (3.8) have corresponding bidegrees with respect to the decomposition $T\mathcal{Z}(\mathcal{F}) = \hat{E} \oplus \hat{F}$, and we see that, if d'' is the $\hat{\mathcal{F}}$ -leafwise differential corresponding to this decomposition, then

$$(3.9) \quad d''\beta = 0, \quad d''\theta = (R_{bcu}^a\beta^c + R_{b\bar{c}u}^a\bar{\beta}^c) \wedge \gamma^u.$$

Accordingly, Proposition 1.1 tells us that \mathcal{C}'_1 is projectable iff

$$(3.10) \quad R_{b\bar{c}u}^a = 0,$$

and \mathcal{C}'_2 is projectable iff

$$(3.11) \quad R_{bcu}^a = 0.$$

Of course, (3.10), (3.11) must hold everywhere on $\mathcal{Z}(\mathcal{F})$, therefore, everywhere on $\mathcal{B}(M, \mathcal{F})$ too (notation of Section 2).

For any given frame $(B, C) \in \mathcal{B}_x(M, \mathcal{F})$ ($x \in M$), we have the following interpretation of the matrix Θ as an $(\text{End } E_x)$ -valued 2-form

$$(3.12) \quad \Theta = pr_S \circ R_{\nabla} \circ pr_{\bar{S}},$$

where $S = \text{span}\{b\}$, $\bar{S} = \text{span}\{\bar{b}\}$ ($B = (b, \bar{b})$). Accordingly (3.10) means that, $\forall x \in M$, $\forall Y \in F_x$, $\forall S^q \subseteq E_x^c$ of real index zero, $\forall V, W \in S$,

$$(3.13) \quad R_{\nabla}(\bar{V}, Y)\bar{W} \in \bar{S}.$$

Indeed, (3.13) implies (3.10) since the latter is (3.13) with arguments belonging to a tangent basis. On the other hand, (3.10) implies (3.13) since, for any given Y, V, W, S there are frames (B, C) such that $S = \text{span}\{b\}$, $V = \alpha b_c$, $W = \beta_1 b_b + \beta_2 b_c$, $Y = \gamma c_u$ ($\alpha, \beta_1, \beta_2 \in \mathbf{C}, \gamma \in \mathbf{R}$). Furthermore, for any given V, W with $\text{span}\{V, W\}$ of real index zero, there are subspaces S_1, S_2 of real index zero such that $S_1 \cap S_2 = \text{span}\{V, W\}$. Hence, (3.13) is equivalent to the fact that, for some coefficients λ, μ ,

$$(3.14) \quad R_{\nabla}(\bar{V}, Y)\bar{W} = \lambda(\bar{V}, Y)\bar{W} + \mu(\bar{W}, Y)\bar{V},$$

holds whenever $\text{span}\{V, W\}$ has real index zero.

Now, notice that $\text{span}\{V, W\}$ is of real index zero iff either $V = 0, W = 0$, or V, W are both proportional to some $X + \sqrt{-1}Z$ where $X, Z \in E$ are \mathbf{R} -independent, or V, W are \mathbf{C} -independent, and their real and imaginary parts are four \mathbf{R} -independent vectors. (Use the fact that the real index is zero iff the real dimension of $\text{span}\{V, \bar{V}, W, \bar{W}\}$ is 4 in the first case, and 8 in the second case.) Therefore, if $\text{span}\{V, W\}$ is of real index zero, the same holds if one or both vectors V, W are replaced by their complex conjugates.

Accordingly, if $q \geq 2$, and if (3.14) holds for V, W , (3.14) also holds for $(\bar{V}, W), (V, \bar{W}), (\bar{V}, \bar{W})$, therefore, (3.14) holds iff the condition (3.6)₁ holds.

In the case $q = 1$, (3.14) reduces to

$$(3.15) \quad R_{\nabla}(V, Y)V = \nu(V, Y)V$$

for any $V = X + \sqrt{-1}Z$ with independent $X, Z \in E_x$, and for some coefficient ν . Separating the imaginary and real parts, (3.15) becomes

$$(3.16) \quad R_{\nabla}(X, Y)Z + R_{\nabla}(Z, Y)X = \nu(X, Y)Z + \nu(Z, Y)X,$$

$$(3.17) \quad R_{\nabla}(X, Y)X - R_{\nabla}(Z, Y)Z = \nu(X, Y)X - \nu(Z, Y)Z,$$

and (3.17) follows from (3.16) by taking $X = Z$. Finally, (3.16) is just the polarization of the E -version of (3.6)₂ with respect to X .

The similar analysis of (3.11) requires the replacement of \bar{V} by V in (3.13), and using the same argument as for (3.14), the projectability condition of \mathcal{C}_2 will be

$$(3.18) \quad R_{\nabla}(V, Y)\bar{W} = \alpha(V, Y)\bar{W},$$

where the real index of $\text{span}\{V, W\}$ is zero, and α is a corresponding coefficient. No term in \bar{V} appears in the right hand side since \bar{V} does not appear in the left hand side of (3.18).

Now, for $q \geq 2$, again, we may replace the pair (V, W) by any of the pairs $(\bar{V}, W), (V, \bar{W}), (\bar{V}, \bar{W})$, and we see that (3.18) is equivalent to

$$(3.19) \quad R_{\nabla}(X, Y)Z = \alpha(X, Y)Z,$$

where $X, Z \in E_x, Y \in F_x, x \in M$.

If $q = 1$, we have to use (3.18) for $W = V = X + \sqrt{-1}Z$, with independent vectors X, Z , and the result is equivalent to the conditions

$$(3.20) \quad R_{\nabla}(X, Y)Z - R_{\nabla}(Z, Y)X = \alpha(X, Y)Z - \alpha(Z, Y)X,$$

$$(3.21) \quad R_{\nabla}(X, Y)X + R_{\nabla}(Z, Y)Z = \alpha(X, Y)X + \alpha(Z, Y)Z.$$

But, (3.21) holds iff $R_{\nabla}(X, Y)X = \alpha(X, Y)X$, which is equivalent to (3.16) by polarization, and the pair of conditions (3.16), (3.20) is equivalent to (3.19). Thus, the only projectability condition is (3.19) again, and we have proven (3.6)₃. Since (3.19) is (3.6)₁ for $\mu = 0$, we see that the projectability of \mathcal{C}_2 implies the projectability of \mathcal{C}_1 , if $q \geq 2$. The same is true for $q = 1$ since (3.19) implies (3.6)₂.

Hence, the condition of simultaneous projectability of $\mathcal{C}_1, \mathcal{C}_2$ is condition (3.6)₃ of the projectability of \mathcal{C}_2 .

Now, to end the proof of Theorem 3.1, it remains to justify the expressions of the coefficients $\lambda, \mu, \nu, \alpha$ as traces. Using the following hints, these are easy consequences of the initial formulas (3.6). For (3.6')₁, (3.6'')₁, compute the traces of the operators which send Z , respectively X , to the vector defined by the E -version of (3.6)₁ then, use (3.5), (3.5''). For (3.6')₂, polarize the E -version of (3.6)₂ with respect to X before computing traces.

Notice that if the splitting $TM = E \oplus F$ is chosen $\lambda, \mu, \nu, \alpha$ may be seen as forms of type $(1, 1)$ on (M, \mathcal{F}) . Otherwise, we may see them as cross sections of $Hom(TM \otimes F, \mathbf{R})$ which vanish if the first argument is in F . Q.e.d.

Clearly, if ∇ is an \mathcal{F} -projectable connection, the almost complex structures $\mathcal{I}_1, \mathcal{I}_2$ will be $\hat{\mathcal{F}}$ -projectable. An interpretation of the simultaneous projectability condition (3.6)₃ is given by

3.2 Proposition. *The almost complex structure \mathcal{I}_2 (hence, \mathcal{I}_1 too), is projectable iff there exists an open covering $M = \cup U_{\alpha}$ endowed with projectable connections ∇^{α} of $N\mathcal{F}/U_{\alpha}$ such that ∇/U_{α} and ∇^{α} are related by a semisymmetric transformation.*

Proof. The structure equations (3.4) show that, locally,

$$(3.22) \quad R_{\nabla}(X, Y) = d''\varpi_M(X, Y),$$

where ϖ_M is the pullback of ϖ to M by local cross sections of $\mathcal{B}(\mathcal{F})$. Hence, locally, (3.6)₃ may be seen as

$$(3.23) \quad d''\varpi_M = \alpha \otimes Id = d''\sigma \otimes Id \quad (\alpha = d''\sigma)$$

(the first equality (3.23) implies $d''\alpha = 0$; then, see [14] for the d'' -Poincaré lemma.) Accordingly, the local connections defined on $N\mathcal{F}$ by

$$(3.24) \quad \theta = \varpi_M - \sigma \otimes Id$$

are projectable. On the other hand, (3.24) exactly is what the older books on differential geometry used to call a *semisymmetric transformation* (the notion was used when one of the two connections was torsionless, which we do not ask here). Equivalently, it is easy to understand that two connections on $N\mathcal{F}$ are semisymmetrically related iff they define the same horizontal distribution \mathcal{H} (see (2.21)) on the twistor space $\mathcal{Z}(\mathcal{F})$.

The converse follows from

3.3 Proposition. *Two adapted connections ϖ, ϖ' of $N\mathcal{F}$ define the same transverse almost complex structures $\mathcal{I}_1, \mathcal{I}_2$ of $\hat{\mathcal{F}}$ iff they are related by a semisymmetric transformation.*

Proof. The local equations of the distributions (3.1), (3.2) of the two connections contain the same forms γ, β , and forms θ, θ' which are related by (2.23) with the horizontal term ζ . Since the connections are adapted, ζ has no term in γ (notation of Proposition 2.1), and if we want $\mathcal{C}_1 = \mathcal{C}'_1$, $\mathcal{C}_2 = \mathcal{C}'_2$, ζ also cannot have terms in $\beta, \bar{\beta}$. Hence, the almost complex structures defined by the two connections coincide iff $\theta' = \theta$. With the same notation as in (3.12), this condition has the equivalent form

$$(3.25) \quad pr_S \circ (\varpi'_M - \varpi)_M(X) \circ pr_{\bar{S}} = 0, \quad X \in TM,$$

and arguments similar to those in the proof of Theorem 3.1 show that (3.25) is equivalent to

$$(3.26) \quad \varpi' = \varpi + \tau \otimes Id,$$

where τ is a 1-form which vanishes on the leaves of \mathcal{F} . Q.e.d.

It is also worthwhile noticing

3.4 Proposition. *A torsionless adapted connection ϖ defines a projectable almost complex structure \mathcal{I}_2 (therefore, \mathcal{I}_1 too) iff ϖ is a projectable connection.*

Proof. By inserting $(3.6')_3$ in (3.5), and taking the trace, it follows that $\alpha = 0$. Since $(3.6)_3$ is locally equivalent to (3.23), ϖ must be a foliated form. Q.e.d.

As a matter of fact, the result is true under the weaker hypothesis $\nabla_Y T_\nabla = 0, \forall Y \in F$.

On the other hand, the projectability of \mathcal{I}_1 alone does not imply the projectability of ϖ . Indeed, if $(3.6)_1$ is inserted in (3.5) we only get $\lambda = \mu$ i.e., if $q \geq 2$, a torsionless adapted connection (or one with $\nabla_Y T_\nabla = 0$) defines a projectable structure \mathcal{I}_1 iff its curvature operator is

$$(3.27) \quad R_\nabla(X, Y)Z = \lambda(X, Y)Z + \lambda(Z, Y)X, \quad Y \in F, X, Z \in E,$$

where

$$(3.28) \quad \lambda(X, Y) = \frac{1}{2q+1} \text{tr} R_\nabla(X, Y).$$

By polarizing the E -version of $(3.6)_2$ with respect to X , and using (3.5), we see that the same is true for $q = 1$.

But, if we add the hypothesis that ∇ induces a projectable connection in $\wedge^{2q} N\mathcal{F}$, which is equivalent to $\text{tr} R_\nabla(X, Y) = 0$ for $Y \in F, X \in E$, we see that ∇ itself must be a projectable connection.

We end this section by

3.5 Proposition. *On $\mathcal{Z}(\mathcal{F})$, the horizontal distribution \mathcal{H} defined by (2.21) is integrable iff ϖ is locally semisymmetric flat.*

Proof. The integrability of \mathcal{H} means $d\theta = d\bar{\theta} = 0 \pmod{\theta, \bar{\theta}}$, and (3.4) tells us that this happens iff $\Theta = 0$. By treating this last condition as we did with (3.12), the integrability condition becomes

$$(3.29) \quad R_\nabla(X, Y) = \lambda(X, Y)Id, \quad X, Y \in \Gamma TM,$$

where λ is a 2-form on M which is closed because of the Bianchi identity. Thus, locally, $\lambda = d\varphi$ for a 1-form φ , and (3.29) holds iff the local connections $\varpi_M - \varphi \otimes Id$ are flat. Q.e.d.

3.6 Corollary. *The expression*

$$(3.30) \quad \Xi := \sqrt{-1} \text{tr}(\theta \wedge \bar{\theta})$$

pulls back to a well defined, real, global 2-form of rank $2q^2$ on $\mathcal{Z}(\mathcal{F})$, which is closed iff ϖ is a locally semisymmetric flat connection.

Proof. That Ξ is well defined follows from (2.17). Then,

$$(3.31) \quad d\Xi = \sqrt{-1}(\Theta \wedge \bar{\theta} - \theta \wedge \bar{\Theta}).$$

Thus, $\Theta = 0$ implies $d\Xi = 0$. The converse holds since $\ker \Xi = \mathcal{H}$, and the kernel of a closed 2-form is involutive. Q.e.d.

4 Existence of suitable connections

Let (M, \mathcal{F}) be a foliated manifold as in the previous sections. If an adapted connection ϖ of $N\mathcal{F}$ defines a foliated almost complex structure \mathcal{I}_1 on the transversal twistor space $\mathcal{Z}\mathcal{F}$ we will say that ϖ is a *twistor-suitable connection*. If ϖ is such that \mathcal{I}_2 (hence, \mathcal{I}_1 too) is projectable, we will say that ϖ is a *strongly twistor-suitable connection*. In this section, we study the existence of suitable connections.

4.1 Proposition. *Let \mathcal{F} be an even-codimensional foliation of a manifold M . Then, $N\mathcal{F}$ has a strongly twistor-suitable connection iff the Atiyah class $\alpha(\mathcal{F}) \in H^1(M, \underline{\wedge^{1,0}M} \otimes \underline{End\,N\mathcal{F}})$ is in the image of $H^1(M, \underline{\wedge^{1,0}M})$ by the mapping $\otimes Id$.*

Proof. We recall that the Atiyah class is the obstruction to the existence of a projectable connection on $N\mathcal{F}$, and send to [7, 8] for details. Underlining in the formulas of the proposition means that we take the corresponding sheaf of *foliated germs*. $\wedge^{1,0}M$ is the space of differential forms of the \mathcal{F} -type $(1, 0)$.

A representative differential form of $\alpha(\mathcal{F})$ is given by the part of type $(1, 1)$ of the curvature of any adapted connection σ on $N\mathcal{F}$ [7]. Thus, if the required connection ϖ exists, and we take $\sigma = \varpi$, the projectability condition $(3.6)_3$ (see also (3.23)) implies the condition of Proposition 4.1.

Conversely, and with the notation of formula (3.23), the condition of Proposition 4.1 means

$$(4.1) \quad d''\sigma_{U_\alpha} = \alpha \otimes Id + d''\eta,$$

where $M = \cup_\alpha U_\alpha$ (U_α open in M), α is a global d'' -closed form of \mathcal{F} -type $(1, 1)$, and η is a global $(End\,N\mathcal{F})$ -valued form of \mathcal{F} -type $(1, 0)$. Then, $\varpi = \sigma - \eta$ is a new adapted connection which defines projectable structures $\mathcal{I}_1, \mathcal{I}_2$ since it satisfies condition $(3.6)_3$. Q.e.d.

For a more geometric answer, we need the following preparations which apply some classical notions to adapted connections of $N\mathcal{F}$.

Let ∇^1, ∇^2 be the covariant derivatives of two adapted connections of $N\mathcal{F}$. Then, there exists a *difference tensor*

$$(4.2) \quad S_x(X, Y) := \nabla_X^2(\pi_N \tilde{Y}) - \nabla_X^1(\pi_N \tilde{Y}) \quad (X, Y \in T_x M, x \in M),$$

where \tilde{Y} is a vector field such that $\tilde{Y}(x) = Y$. In particular, by using a foliated field \tilde{Y} , we see that S vanishes if any of its arguments is in F . Formula (3.24) tells that ∇^1, ∇^2 are semisymmetrically related iff

$$(4.3) \quad S = \sigma \otimes \pi_N$$

for some scalar $(1, 0)$ -form σ on (M, \mathcal{F}) .

On the other hand, we need the *transposed connection* of a connection ∇ , defined by

$$(4.4) \quad {}^t\nabla_X(\pi_N Y) = \nabla_Y(\pi_N \tilde{X}) + \pi_N[\tilde{X}, Y],$$

where $\tilde{X}(x) = X \in T_x M$, $x \in M$, and \tilde{X} is foliated. Using ${}^t\nabla$, we get the torsionless connection

$$(4.5) \quad {}^s\nabla_X(\pi_N Y) = \frac{1}{2}(\nabla_X(\pi_N Y) + {}^t\nabla_X(\pi_N Y)),$$

called the *symmetric part* of ∇ .

Furthermore, two torsionless adapted connections ∇^1, ∇^2 of $N\mathcal{F}$ are said to be *projectively related* if their difference tensor is of the form

$$(4.6) \quad S = \phi \odot \pi_N,$$

where ϕ is a $(1, 0)$ -form on M and \odot denotes the symmetric tensor product. The name comes from the fact that, on manifolds (i.e., \mathcal{F} is the foliation by points), two such connections have the same selfparallel lines (e.g., [5]). Accordingly, a foliation \mathcal{F} is called *transversally projective* [10, 16] if it is endowed with the following data: i) an open covering $M = \cup_\alpha U_\alpha$ by \mathcal{F} -adapted coordinate neighborhoods, ii) a family of foliated connections ∇^α on $N\mathcal{F}/U_\alpha$ which are projectively related over $U_\alpha \cap U_\beta$ i.e.,

$$(4.7) \quad S_{\alpha\beta} := \nabla^\beta - \nabla^\alpha = \phi_{\alpha\beta} \odot \pi_N \quad (\phi_{\alpha\beta} \in \wedge_{fol}^1(U_\alpha \cap U_\beta)),$$

where the index *fol* means foliated forms. If maximal, this data system defines a *transversal projective structure* of \mathcal{F} . In the framework of Čech cohomology, $\{\phi_{\alpha\beta}\}$ defines a cohomology class $[\phi] \in H^1(M, \underline{\wedge}^{1,0}M)$ which we call the *complementary class* of the projective structure. If we put

$$(4.8) \quad \phi_{\alpha\beta} = \psi_\alpha - \psi_\beta,$$

where ψ_α, ψ_β are transverse 1-forms on U_α, U_β , we get the global adapted connection

$$(4.9) \quad \nabla = \nabla^\alpha + \psi_\alpha \odot \pi_N,$$

which is projectively related to the local connections ∇^α but, of course, it may not be foliated. A global foliated connection (4.9) can be obtained iff $[\phi] = 0$.

Now, assume that $N\mathcal{F}$ has a strongly suitable connection ϖ , and let $\{\nabla^\alpha\}$ be the associated system of local, foliated connections, semisymmetrically related to ϖ , defined by Proposition 3.2. Then, any two connections of this system are semisymmetrically related, say

$$(4.10) \quad \nabla^\beta = \nabla^\alpha + 2\phi_{\alpha\beta} \otimes \pi_N,$$

and it follows that the symmetric connections ${}^s\nabla^\alpha$ satisfy (4.7), and define a transversal projective structure of \mathcal{F} .

Conversely, assume that \mathcal{F} is a *transversally projective foliation*, with the structure defined by the data of (4.7). Then, the connections

$$(4.11) \quad \tilde{\nabla}^\alpha = \nabla^\alpha + \frac{1}{2}\tau_\alpha,$$

where ∇_α are those of (4.7), have the torsion tensor τ_α , and for these connections one has the transition relations

$$(4.12) \quad \tilde{\nabla}^\beta = \tilde{\nabla}^\alpha + \phi_{\alpha\beta} \odot \pi_N + \xi_{\alpha\beta}, \quad \xi_{\alpha\beta} := \frac{1}{2}(\tau_\alpha - \tau_\beta).$$

We would like to be able to choose τ such that (4.12) would be a semisymmetric transformation of foliated transversal connections. This happens iff the local tensors τ_α are foliated, and

$$(4.13) \quad \xi_{\alpha\beta} = \pi_N \wedge \phi_{\alpha\beta}.$$

The right hand side of the equality (4.13) defines a Čech cohomology class $[\phi]_N \in H^1(M, \underline{\wedge^{2,0}M} \otimes N\mathcal{F})$, which we call the *normal complementary class* of the projective structure. Then, clearly, we can get the required situation of foliated, semisymmetric relations iff $[\phi]_N = 0$. Therefore, we have proven

4.2 Theorem. *Let \mathcal{F} be an even-codimensional foliation of a manifold M . Then, $N\mathcal{F}$ has an adapted connection which defines projectable, almost complex structures $\mathcal{I}_1, \mathcal{I}_2$ of $N\hat{\mathcal{F}}$ on $\mathcal{Z}(\mathcal{F})$ iff \mathcal{F} has a transversal, projectable, projective structure, with a vanishing normal complementary class.*

The existence condition of (not strongly) suitable connections is weaker. Namely, we have

4.3 Theorem. *If there exists an adapted connection of $N\mathcal{F}$ which produces a projectable almost complex structure \mathcal{I}_1 of $N\hat{\mathcal{F}}$, there also exists a torsionless connection with the same property. A torsionless twistor-suitable connection exists iff \mathcal{F} is a transversally projective foliation.*

Proof. The E -version of the projectability condition $(3.6)_1$ holds pointwisely iff it holds for any vector field $Y \in \Gamma F$, and any projectable vector fields $X, Z \in \Gamma E$. Then, if we express R_∇ by covariant derivatives and use Bott's condition

$$\nabla_Y X = \pi_E[Y, X], \quad Y \in \Gamma F, X \in \Gamma E,$$

(see the $N\mathcal{F}$ -version of this condition in [2]), we get the following equivalent form of $(3.6)_1$

$$(4.14) \quad \pi_E[Y, \nabla_X Z] = -\lambda(X, Y)Z - \mu(Z, Y)X,$$

for all $Y \in \Gamma F$, and for all projectable $X, Z \in \Gamma E$.

Since from the Bott condition and the projectability of X, Z we get

$$\pi_E[Y, \pi_E[X, Z]] = \nabla_Y \pi_E[X, Z] = 0,$$

the E -version of (4.5) yields

$$\pi_N[Y, {}^s \nabla_X Z] = \frac{1}{2} \pi_E([Y, \nabla_X Z] + [Y, \nabla_Z X]),$$

and (4.14) implies that the symmetric part of ∇ satisfies (3.27) with $(\lambda + \mu)/2$ in the role of λ .

This proves the first conclusion for $q \geq 2$. The proof for $q = 1$ is similar but, starting with $(3.6)_2$ which is equivalent to

$$(4.15) \quad \pi_E[Y, \nabla_X X] = -\nu(X, Y)X,$$

for all $Y \in \Gamma F$, and for all foliated $X \in \Gamma E$.

For the second assertion of Theorem 4.3, if \mathcal{F} is transversally projective, it has the connection ∇ of (4.9) where ∇^α are foliated connections. From the E -version of (4.9), it follows that ∇ satisfies (4.14) and (4.15) for $\lambda = \mu = d''\psi_\alpha$. λ is a global form since the differences $\psi_\alpha - \psi_\beta$ are foliated 1-forms.

Conversely, if ∇ is a twistor-suitable, torsionless connection, we have (3.27), (3.28) where $d''\lambda = 0$ hence, $\lambda = d''\sigma$, σ being a local form of \mathcal{F} -type $(1, 0)$. Then, we may locally write (3.27) as

$$(4.16) \quad \nabla_Y \nabla_X Z = (Y\sigma(X))Z + (Y\sigma(Z))X,$$

where $Y \in \Gamma F$ and X, Z are foliated cross sections of E . From (4.16), it follows that the local projectively related connection

$$\nabla' = \nabla - \sigma \odot \pi_N$$

is foliated. Q.e.d.

Notice that any connection obtained from ∇ of (4.9) by a projective transformation also yields a projectable structure \mathcal{I}_1 . But, the structures \mathcal{I}_1 of the different connections are different. Indeed, the form ϕ of a projective difference (4.6) of connections is real hence, θ of (3.1) changes by an additional term which contains both β and $\bar{\beta}$.

Furthermore, if \mathcal{F} is a transversally projective foliation with a vanishing normal complementary class, the connections which it generates by the procedures of Theorems 4.2, 4.3 may not be semisymmetrically related hence, produce different structures $\mathcal{I}_1, \mathcal{I}_2$.

4.4 Remark. *In the same way as for Proposition 4.1, we can see that a foliation \mathcal{F} has a torsionless twistor-suitable connection (equivalently, it has a projectable transversal projective structure) iff the Atiyah class of \mathcal{F} is in the image of the mapping*

$$\odot Id : H^1(M, \underline{\wedge^{1,0} M}) \rightarrow H^1(M, \underline{\wedge^{1,0} M} \otimes \underline{End N\mathcal{F}}).$$

A way of looking for examples could be the use of a Riemannian metric g on (M, \mathcal{F}) . Then, we will take E orthogonal to F and there exists a unique adapted torsionless connection ∇ such that the ∇ -parallelism along paths tangent to E preserves the metric $g|_E$. Namely, $\nabla_Y X = \pi_E[Y, X]$ if $Y \in \Gamma F, X \in \Gamma E$, and $\nabla_Y X$ is determined like the Levi-Civita connection [6] if $Y, X \in \Gamma E$. The result is the E -component of the *second connection* of (M, g, \mathcal{F}) [13, 14], and we call ∇ the *canonical connection* of (M, \mathcal{F}, g) . If the transversal projective structure defined by the canonical connection ∇ is projectable, the foliation $\hat{\mathcal{F}}$ of $\mathcal{Z}(\mathcal{F})$ will have a projectable, almost complex structure \mathcal{I}_1 . But, \mathcal{I}_2 will be projectable iff ∇ is projectable (Proposition 3.4) hence, \mathcal{F} is a Riemannian foliation with the *bundle-like metric* g [8].

5 The integrability conditions

Now, in the cases where projectability holds we will discuss the integrability conditions of the structures $\mathcal{I}_1, \mathcal{I}_2$ defined in Section 3 i.e., the conditions which ensure that the local, almost complex slice spaces of $(\mathcal{Z}(\mathcal{F}), \hat{\mathcal{F}})$ are, in fact, complex manifolds.

The integrability condition is equivalent to the fact that the local equations of $\hat{\mathcal{F}}$ can be put under the form $z^\sigma = \text{const.}, \bar{z}^\sigma = \text{const.}$, where z^σ are complex local coordinates with complex analytic transition functions. Equivalently, the distribution \mathcal{C}'_1 or \mathcal{C}'_2 defined by (3.1'), (3.2') must be *Nirenberg integrable*, which (by a theorem of Nirenberg [9], and since $\mathcal{C}'_a + \bar{\mathcal{C}}'_a = T\mathcal{Z}(\mathcal{F})$ ($a = 1, 2$)) happens iff \mathcal{C}'_1 or \mathcal{C}'_2 , respectively, is an involutive distribution.

Accordingly, the structure equations (3.3), (3.4) tell us that the integrability conditions of \mathcal{I}_1 are

$$(5.1) \quad d\beta = -\omega \wedge \beta - \theta \wedge \bar{\beta} + T = 0 \quad (\text{mod. } \beta, \theta),$$

$$(5.2) \quad d\theta = -\omega \wedge \theta - \theta \wedge \bar{\omega} + \Theta = 0 \quad (\text{mod. } \beta, \theta),$$

The integrability conditions of \mathcal{I}_2 are given by (5.1) modulo $\beta, \bar{\theta}$ and (5.2) modulo $\bar{\beta}, \theta$.

5.1 Theorem. *Let M be a foliated manifold with the foliation \mathcal{F} of codimension $2q$, and assume that ∇ is a strongly suitable connection on $N\mathcal{F}$.*

Then, \mathcal{I}_2 is never integrable, and \mathcal{I}_1 is integrable iff either i) $q \geq 3$ and the torsion and curvature of ∇ are of the following form

$$(5.3) \quad T_{\nabla}(X, Y) = (\alpha \wedge \pi_N)(X, Y) = \alpha(X)\pi_N(Y) - \alpha(Y)\pi_N(X),$$

$$(5.4) \quad R_{\nabla}(X, Y)(\pi_N Z) = \beta(X, Z)(\pi_N Y) - \beta(Y, Z)(\pi_N X) + \gamma(X, Y)(\pi_N Z),$$

or ii) $q = 1$, or iii) $q = 2$, the torsion is given by (5.3), and

$$(5.5) \quad R_{\nabla}(X, Y)\pi_N X = \xi(X, Y)\pi_N X + \zeta(X, X)\pi_N Y,$$

or, equivalently by polarization,

$$(5.5') \quad \begin{aligned} & R_{\nabla}(X, Y)\pi_N Z + R_{\nabla}(Z, Y)\pi_N X \\ &= \xi(X, Y)\pi_N Z + \xi(Z, Y)\pi_N X + 2\zeta(X, Z)\pi_N Y. \end{aligned}$$

In these formulas, X, Y, Z are vector fields on M , α is a 1-form which vanishes on F , γ is a 2-form which satisfies $\gamma(X, Y) = \alpha(X, Y)$ with α of (3.6)₃ if $Y \in F$, β is a 2-covariant tensor field which vanishes if one of its arguments is in F , ξ is a two-form such that $\xi(X, Y) = \alpha(X, Y)$ with α of (3.6)₃ if $Y \in F$, and ζ is a covariant symmetric tensor field which vanishes if one of its arguments is in F . Moreover, after a choice of the transversal bundle E , the values of $\alpha, \beta, \gamma, \xi, \zeta$ are completely determined by traces of T and R .

Proof. Equation (5.1) never holds modulo $\beta, \bar{\theta}$ hence, \mathcal{I}_2 is never integrable, and we only have to discuss the integrability of \mathcal{I}_1 .

The conclusions could be justified as follows. By Proposition 3.2, it suffices to write down the integrability conditions of the structure \mathcal{I}_1 of the local projectable connections ∇^α , semisymmetrically related with ∇ . But, on arguments in $N\mathcal{F}$, these conditions are the same as after projection on the local slice spaces i.e., conditions for twistor spaces on manifolds, where they exactly are (5.3), (5.4), (5.5) for ∇^α [11]. Then, it is enough to show that the conditions are invariant by a semisymmetric transformation of the connection, which is easy.

However, we prefer to give a different, elementary proof. First, we fix the bundle E , and identify the projections π_N and π_E . This will produce the values of the coefficients $\alpha, \beta, \gamma, \xi, \zeta$ on arguments in E , and, then, the

addition of the conditions stated by the theorem for these coefficients will allow us to extend $\alpha, \beta, \gamma, \xi, \zeta$ to arguments in F too.

Equation (5.1) provides a *torsion integrability condition*, and (5.2) provides a *curvature integrability condition*. From (3.7), we see that (5.1) holds iff $T_{\bar{b}\bar{c}}^a = 0$ for all the frames in $\mathcal{B}(M, \mathcal{F})$ (notation of Section 2). Using again the notation of the proof of Theorem 3.1, we see that the meaning of the previous condition is that $\forall S^q \subseteq E^c$ of real index zero and $\forall V, W \in S$, one must have

$$(5.6) \quad T_{\nabla}(\bar{V}, \bar{W}) \in \bar{S}.$$

As in the proof of Theorem 3.1, if $q \geq 2$, for any independent vectors $V, W \in E^c$ with $L := \text{span}\{V, W\}$ of real index zero, we may write L as $L = S_1 \cap S_2$ for some S_1, S_2 as in (5.6). Therefore, after a conjugation, (5.6) holds iff

$$(5.7) \quad T_{\nabla}(V, W) = a(V)W + b(W)V.$$

Then, because of the skew symmetry, (5.7) becomes (5.3).

Now, from (3.8), and since (3.10) is supposed to hold, we see that the curvature integrability condition (5.2) means $R_{\bar{b}\bar{c}\bar{d}}^a = 0$, and, with the same notation as above, this is equivalent to $R_{\nabla}(\bar{V}, \bar{W})(\bar{U}) \in \bar{S}$, whenever the space $L := \text{span}\{V, W, U\}$ has real index zero. If $q \geq 3$, and seeing L as $L = \bar{S}_1 \cap \bar{S}_2$, the integrability condition becomes

$$(5.8) \quad R_{\nabla}(V, W)U = \gamma(V, W)U + \beta(V, U)W + \eta(W, U)V.$$

Using the skew symmetry in V, W , we may write condition (5.8) under the form (5.4).

For $q = 1$, the integrability conditions are satisfied for any strongly suitable connection.

For $q = 2$, (5.3) continues to be the necessary and sufficient torsion integrability condition (as we saw above). Furthermore, (5.4) continues to be a sufficient curvature integrability condition but, not a necessary one. Indeed, if $q = 2$, the curvature integrability condition is that for all independent $V, W \in E^c$ which span a space of real index zero one has

$$(5.9) \quad R_{\nabla}(V, W)(\lambda V + \mu W) = \sigma V + \tau W$$

$(\lambda, \mu, \sigma, \tau \in \mathbf{C})$. For (5.9) to hold, it is enough that

$$(5.10) \quad R_{\nabla}(V, W)V = \alpha(V, W)V + \beta(V, V)W.$$

(Of course, in (5.9), (5.10) all the coefficients are new.) Since the condition put on V, W is equivalent with the fact that the real and imaginary parts $(Re V, Im V, Re W, Im W)$ are \mathbf{R} -independent, we may replace them by the conjugate vectors as well, and (5.10) also provides the necessary and sufficient curvature integrability condition in terms of real vectors of TM . Namely, we exactly obtain condition (5.5).

Furthermore, computing with (5.3) the trace of the well defined operator $i(X)T_{\nabla} : N\mathcal{F} \rightarrow N\mathcal{F}$, $X \in \Gamma TM$, we get

$$(5.11) \quad \alpha(X) = \frac{1}{2q-1} tr i(X)T_{\nabla}.$$

Similarly, from (5.4) we get

$$(5.12) \quad tr R_{\nabla}(X, Y) = 2[alt(\beta)(X, Y) + q\gamma(X, Y)],$$

where $alt(\beta)$ is the skew-symmetric part of β , valid for all $X, Y \in \Gamma TM$ because of the conditions on β, γ , and then, with the fixed bundle E ,

$$(5.13) \quad Ric_{(\nabla, E)}(X, Y) = -(2q-1)\beta(X, Y) - \gamma(X, Y),$$

valid $\forall X, Y \in \Gamma E$. By symmetrization, (5.13) gives

$$(5.14) \quad sym(\beta)(X, Y) = -\frac{1}{2q-1} sym(Ric_{(\nabla, E)})(X, Y), \quad X, Y \in \Gamma E,$$

where sym denotes the symmetric part of a tensor. Furthermore, the alternation of (5.13) together with (5.12) form an equation system which yields

$$(5.15') \quad \gamma(X, Y) = \frac{1}{2q^2 - q - 1} [alt(Ric_{(\nabla, E)})(X, Y) + \frac{2q-1}{2} tr R_{\nabla}(X, Y)], \quad X, Y \in \Gamma E,$$

$$(5.15'') \quad alt \beta(X, Y) = -\frac{1}{2q^2 - q - 1} [q alt Ric_{(\nabla, E)}(X, Y) + \frac{1}{2} tr R_{\nabla}(X, Y)].$$

Finally, the value of β is provided by inserting the values (5.15'), (5.15'') into $\beta = sym(\beta) + alt(\beta)$.

For the case $q = 2$, we compute traces in the E -version of (5.5') and, first, $\forall X, Y \in \Gamma E$, we get

$$(5.16) \quad tr R_{\nabla}(X, Y) + Ric_{(\nabla, E)}(Y, X) = 5\xi(X, Y) + 2\zeta(X, Y).$$

Then, we alternate and symmetrize, and get

$$(5.17) \quad \xi(X, Y) = \frac{1}{5}[tr R_{\nabla}(X, Y) - alt(Ric_{(\nabla, E)})(X, Y)],$$

$$(5.18) \quad \zeta = \frac{1}{2}sym(Ric_{(\nabla, E)})(X, Y), \quad X, Y \in \Gamma E.$$

Q.e.d.

Now, let us discuss the integrability conditions (5.1), (5.2) of the projectable structure \mathcal{I}_1 defined by an adapted torsionless connection ∇ which is projectively related to a given transversal projective structure of the foliation \mathcal{F} (Theorem 4.3).

Condition (5.1) holds since $T = 0$. In (5.2), Θ has one more term than in the case of Theorem 5.1 but, this is a term in $\beta^c \wedge \gamma^u$ (see (3.8)). Therefore, the integrability condition (5.2) again means $R_{\bar{b}\bar{c}\bar{d}}^a = 0$, as in Theorem 5.1. Hence, the only integrability condition for \mathcal{I}_1 is again (5.4), if $q \geq 3$, and (5.5), if $q = 2$.

Furthermore, since there is no torsion, we have the transversal Bianchi identity

$$(5.19) \quad \sum_{Cycl(X, Y, Z)} R_{\nabla}(X, Y)Z = 0, \quad X, Y, Z \in \Gamma E,$$

and, if we insert the value of R_{∇} as given by the E -version of (5.4) in (5.19), and take the trace, we obtain $\gamma(X, Y) = \beta(X, Y) - \beta(Y, X)$, $\forall X, Y \in \Gamma E$.

The final result can be expressed as follows

5.2 Theorem. *Let \mathcal{F} be a transversally projective foliation of codimension $2q$, $q \geq 3$. Then its family of global, adapted, projectively equivalent, torsionless connections of $N\mathcal{F}$ defines a family of projectable structures \mathcal{I}_1 which are simultaneously integrable or not. Integrability occurs iff*

$$(5.20) \quad R_{\nabla}(X, Y)\pi_N Z = \beta(X, Z)\pi_N Y - \beta(Y, Z)\pi_N X$$

$$+(\beta(X, Y) - \beta(Y, X))\pi_N Z, \quad X, Y, Z \in \Gamma TM,$$

where necessarily, $\beta(X, Y) = 0$ if $Y \in \Gamma F$, $\beta(Y, X) = -\lambda(X, Y)$, with λ of (3.28), if $Y \in \Gamma F$, and for any fixed E , $\forall X, Y \in \Gamma E$,

$$(5.21) \quad \beta(X, Y) = \frac{1}{2q-1} \text{sym}(\text{Ric}_{(\nabla, E)})(X, Y) + \frac{1}{4q+2} \text{tr } R_{\nabla}(X, Y).$$

The meaning of condition (5.20) is that the transversal projective structure of \mathcal{F} is projectively flat. Under the same hypotheses, but for $q = 2$, \mathcal{I}_1 is integrable iff

$$(5.22) \quad R_{\nabla}(X, Y)\pi_N X = \xi(X, Y)\pi_N X \quad X, Y \in \Gamma TM,$$

where ξ is a 2-form which satisfies $\xi(X, Y) = 2\lambda(X, Y)$, with λ of (3.28), if $Y \in \Gamma F$, and if E is fixed,

$$(5.23) \quad \xi(X, Y) = \frac{1}{5} [\text{tr } R_{\nabla}(X, Y) - \text{Ric}_{(\nabla, E)}(X, Y)], \quad X, Y \in \Gamma E.$$

For $q = 1$, integrability always holds.

Proof. Condition (5.20) was already justified, and (5.21) follows by taking the necessary traces in (5.20). Condition (5.22) has a similar justification namely, the use of (5.5') and (5.19) yields $\zeta = 0$ hence, we have proven (5.22) for arguments in ΓE . Then, (5.23) follows from (5.17), (5.18), and $\zeta = 0$.

Finally, the relation which a projective transformation (4.6) of a connection implies on the curvature tensors can be easily deduced, and it is well known [5]. This relation preserves the form (5.20) of the curvature (while changing β of course), and (5.20) holds iff the connection is projectively transformable into a flat connection. This explains all the other assertions of the theorem. Q.e.d.

We may look for examples by first defining the notion of a *transversally homographic foliation* as follows. In \mathbf{R}^s , there exists a pseudogroup of *homographies*, i.e., projective transformations written in non homogeneous coordinates

$$(5.24) \quad \tilde{x}^i = \frac{a_h^i x^h + b^i}{c_h^i x^h + d^i},$$

where $(i, h = 1, \dots, s)$ and the coefficients are real numbers; homographies are defined on open sets of \mathbf{R}^s . A foliation \mathcal{F} of codimension s will be called

transversally homographic if its Haefliger cocycle [8] may be taken in the pseudogroup (5.24).

Since homographies send a line segment onto a line segment, the system of local adapted connections of $N\mathcal{F}$ defined by the flat connection of \mathbf{R}^s consists of projectively related connections. Hence, every transversally homographic foliation is transversally projective flat. Clearly, the converse also holds. Thus, the two notions coincide.

A concrete example is the following one given in [10]. Let G be the connected component of the identity in the general projective group in dimension $s + 1$, H be the affine subgroup of G , which fixes a given hyperplane, and K the subgroup of H which is isomorphic to $SO(s)$. Then $B := G/H$ is diffeomorphic to the real projective space \mathbf{RP}^s , and $V := G/K$ is a bundle over B . The fibers of this bundle define a G -invariant, transversally homographic foliation \mathcal{S} of codimension s . Then, \mathcal{S} induces transversally homographic foliations \mathcal{F} on all the manifolds $M = D \setminus V$ where D is a discrete subgroup of G .

In particular, if $s = 2q$, all the foliations \mathcal{F} obtained in this way have a twistor space $\mathcal{Z}(\mathcal{F})$ with a transversally holomorphic lifted foliation $\hat{\mathcal{F}}$.

Other examples could be provided by foliations \mathcal{F} on Riemannian manifolds (M, g) , such that the canonical connection of the normal bundle (see the end of Section 4) is projectively flat but, we do not have yet concrete examples of such foliations.

6 The case of a Riemannian manifold

In this section we discuss the transversal twistor space $\mathcal{Z}(\mathcal{F})$ of a foliation \mathcal{F} defined on a Riemannian manifold M with the metric g .

Before doing this we notice that a general metric g may not allow for a twistorial construction, in the sense of Section 2, on the bundle $\mathcal{Z}(\mathcal{F}, g)$ of pointwise, g/E -orthogonal, complex structures of E , because $\mathcal{Z}(\mathcal{F}, g)$ may not be a foliated bundle, and then the foliation $\hat{\mathcal{F}}$ does not exist. If \mathcal{F} is a Riemannian foliation with the *bundle-like metric* g , a twistor bundle $\mathcal{Z}(\mathcal{F}, g)$ with a foliation $\hat{\mathcal{F}}$ exists. For this bundle we have

$$(6.1) \quad \mathcal{Z}(\mathcal{F}, g) = \cup \mathcal{Z}(\mathcal{F}/U_\alpha, g_\alpha = g/U_\alpha),$$

where $\{U_\alpha\}$ is a covering of M by \mathcal{F} -adapted coordinate neighborhoods,

and the transversal almost complex structures of $\hat{\mathcal{F}}$ defined by the Levi-Civita connections of $g_E := g|_E$ [8] are the lifts of those of the Riemannian twistor spaces of the slice spaces $(U_\alpha/\mathcal{F} \cap U_\alpha, g_E|_{U_\alpha})$. Hence, all the results of a local character (e.g., integrability) will be just the same as the classical ones e.g., [1, 11, 3]. Moreover, if \mathcal{F} is a *conformal foliation* i.e., only the conformal structure defined by g_E is projectable [10, 15], a space (6.1) where g_α are foliated and conformal to $g_E|_{U_\alpha}$, exists. This is the space of *conformal twistors*, and it has a well defined almost complex structure. Namely, the lift of the conformal invariant, almost complex structure of the local slice spaces $(U_\alpha/\mathcal{F} \cap U_\alpha, g_\alpha)$ [3].

Now, coming back to the space $\mathcal{Z}(\mathcal{F})$, the interesting point to be made is that, using an adapted connection ϖ of \mathcal{F} , g can be lifted to a pseudo-Riemannian metric \hat{g} on $\mathcal{Z}(\mathcal{F})$ as follows. On the principal bundle $\mathcal{B}(M, \mathcal{F})$ of Section 2, we can define the following matrices of globally defined functions

$$(6.2) \quad g_1 = {}^t C \cdot C, \quad g_2 = {}^t b \cdot \bar{b} = {}^t \bar{g}_2 \quad (b, \bar{b}, C) \in \mathcal{B}(M, \mathcal{F}),$$

where the dot is the g -scalar product at $\pi(b, \bar{b}, C) \in M$, and t denotes transposition of matrices. Then, it is an immediate consequence of formulas (2.17), (2.18) that

$$(6.3) \quad \hat{g} := {}^t \gamma \otimes (g_1 \gamma) + {}^t \beta \otimes (g_2 \bar{\beta}) + {}^t \bar{\beta} \otimes ({}^t g_2 \beta) - \text{tr}(\theta \otimes \bar{\theta} + \bar{\theta} \otimes \theta),$$

where the notation is that of Proposition 2.1, pulls back to a well defined pseudo-Riemannian metric of $\mathcal{Z}(\mathcal{F})$ of (\pm) -signs $(p + 2q + q^2, q^2)$.

6.1 Remark. *The metric \hat{g} also exists if g is a pseudo-Riemannian metric which is nondegenerate on F but, the signature will be different.*

It follows from (6.3) that $\hat{g}|_{\hat{E}}$ is compatible with the almost complex structures $\mathcal{I}_1, \mathcal{I}_2$ defined by the same adapted connection ϖ . Accordingly, there exists a real 2-form of $\hat{\mathcal{F}}$ -type $(2, 0)$ on $\mathcal{Z}(\mathcal{F})$, given by

$$(6.4) \quad \Phi = \sqrt{-1} \{ {}^t \beta \wedge (g_2 \bar{\beta}) - \text{tr}(\theta \wedge \bar{\theta}) \},$$

which is nondegenerate on \hat{E} .

6.2 Remark. *For the construction of a 2-form Φ as in (6.4), it suffices to start with a pseudo-Riemannian transversal metric of \mathcal{F} on M .*

6.3 Proposition. *The 2-form Φ is $\hat{\mathcal{F}}$ -projectable iff g is an \mathcal{F} -bundle-like metric on M , and ϖ is a strongly twistor-suitable connection. In particular, if ϖ is the canonical connection of \mathcal{F} , Φ is projectable iff g is a bundle-like metric of \mathcal{F} .*

Proof. From $\hat{\mathcal{F}}$ -type considerations and formula (3.9), it follows that $d''\Phi = 0$ iff each term of the right hand side of (6.4) is d'' -closed. For the second term this happens iff (3.10) and (3.11) hold i.e., ϖ is a strongly suitable connection. For the first term in (6.4), if we put $b_i = \xi_i^a X_a$, where $\{X_a\}$ is a local foliated basis of E , then (ξ_i^a) are $\hat{\mathcal{F}}$ -local transversal coordinates on $\mathcal{Z}(\mathcal{F})$, and d'' -closedness holds iff $g(X_a, X_b)$ are foliated functions on M . Q.e.d.

6.4 Corollary. *If the conditions of Proposition 6.3 hold, \hat{g} is an $\hat{\mathcal{F}}$ -bundle-like metric.*

We may also notice that the forms defined on $\mathcal{B}(\mathcal{F})$ by

$$(6.5) \quad \hat{h} := {}^t\beta \otimes (g_3\beta) + {}^t\bar{\beta} \otimes (\bar{g}_3\bar{\beta}),$$

$$(6.6) \quad \hat{\Psi} := \sqrt{-1} [{}^t\beta \wedge (g_3\beta) + {}^t\bar{\beta} \wedge (\bar{g}_3\bar{\beta})],$$

where $g_3 := {}^tb \cdot b$, descend to well defined global forms on $\mathcal{Z}(\mathcal{F})$. (In fact, each term of (6.5) and (6.6) has this property.) Then, $\hat{g} + \hat{h}$ is again a pseudo-Riemannian metric on $\mathcal{Z}(\mathcal{F})$, whose \mathcal{H} -term is the lift of g , and $\Phi + \Psi$ is an $\hat{\mathcal{F}}$ -transversal almost symplectic structure which is projectable iff the conditions of Proposition 6.3 hold.

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